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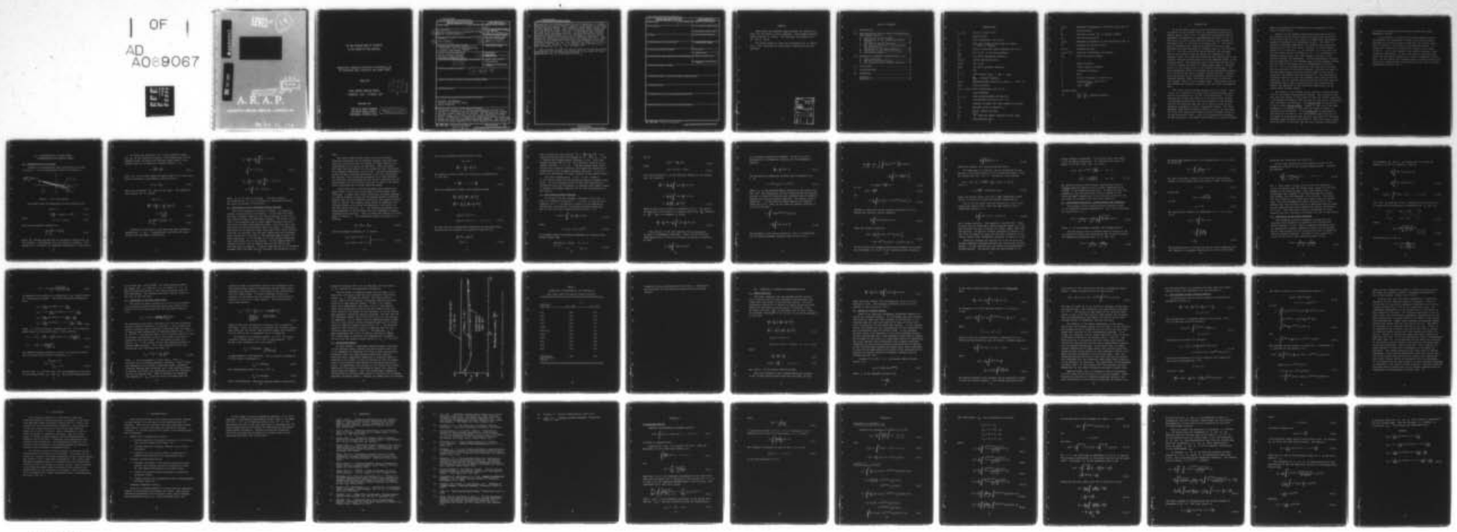

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ON THE SINGULAR ROLE OF VISCOSITY IN THE THEORY OF THIN AIRFOILS

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The general problem of a lifting incompressible viscous thin airfoil is formulated and the viscous counterpart of the classical thin airfoil equation is derived. For any Reynolds number, however large, it is shown that the Cauchy singularity in the kernel is replaced by a logarithmic singularity and the resulting kernel does not have upstream/downstream symmetry. The downstream influence decays algebraically, while upstream influence decays over

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exponentially with distance away from the singularity. A moment solution of the viscous airfoil equation is developed in the limit of high Reynolds number. The classical steady state "Kutta Condition" is derived in the limit and the Reynolds number correction is found to be of order (1/ln Re), much greater than inviscid boundary layer thickness effects. For Reynolds numbers between one and ten million there is a 20% reduction in the lift curve slope that offsets the increase due to geometric thickness. The correction correlates reasonably well with experiment for a large variety of thin airfoils. It is shown that because of the logarithmic dependence on Re the correction is large but the variation with Re is small for Re greater than a million; e.g., only 3% variation of C_{L_α} for $10^6 < Re < 10^7$.

The unsteady viscous thin airfoil equation is derived and the singularities of the kernel are discussed in the light of recent experimental work on oscillating airfoils.

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PREFACE

This report was prepared under Contract No. N00014-76-C-0576, for the Office of Naval Research. The Program Technical Monitor was Mr. M. Cooper. The Principal Investigator was Dr. John E. Yates.

The author wishes to thank his colleagues Drs. G. Sandri and C. duP. Donaldson for many helpful discussions and suggestions.

TABLE OF CONTENTS

NOMENCLATURE

A_o , A_1	see (2.51) and (2.52)
c	airfoil chord
C_L	lift coefficient, see (2.10)
C_{L_α}	lift curve slope at zero angle of attack (/ radian or /degree)
C_M	moment coefficient about midchord, see (2.11)
C_{nm}	see (2.50), (2.53) and Appendix B
div	2-D vector divergence operation
$f(x,t)$	airfoil deflection mode
$F_n(\xi)$	see (2.48)
grad	2-D vector gradient operation
i	$\sqrt{-1}$
$\vec{i}$, $\vec{j}$	unit vectors along x and y axes
k	$\frac{\omega c}{2u_\infty}$, reduced frequency
$K_n(z)$	modified Bessel function of order n , Ref. 19
$K(x)$	kernel function
$L(x)$, $L(x,t)$	lift distribution, see (2.22)
p	local pressure
p_∞ , ρ_∞	free stream pressure and density
P	dimensionless pressure, see (2.8)
$P^\pm$	pressure on upper and lower surface of airfoil
$\|q\|$	absolute value of quantity q
q_∞	$\frac{1}{2}\rho_\infty u_\infty^2$ dynamic pressure
Re	$\frac{u_\infty c}{\nu}$ Reynolds number referred to full chord
t	dimensionless time

$T_n(x)$	Chebyshev polynominal of the first kind, Ref. 19
$\vec{u}$	fluid velocity
$\vec{u}_s$	velocity of point $\vec{x}_s$ on airfoil surface
u_∞	free stream velocity
$U_n(x)$	Chebyshev polynominal of the second kind, Ref. 19
$\vec{v}$	perturbation velocity field
$w(x,t)$	downwash velocity
(x,y) or $\vec{x}$	Cartesian coordinates, see figure 1
$\vec{x}_s$	point on airfoil surface
α	angle of attack
γ	.57722, Euler's constant
ν	kinematic viscosity
σ	$Re/4$
τ	airfoil thickness to chord ratio
ψ	streamfunction, see (2.15)
Ω	$\sigma\left(1 - \frac{2ik}{\sigma}\right)^{1/2}$

Special symbol

∇^2	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, Laplace operation

I. INTRODUCTION

In recent years there has been considerable interest in the effect of the boundary layer in unsteady and steay aerodynamic theory, Refs. 1-13. A definitive oscillating airfoil experiment is described by Satyanarayana and Davis in Ref. 14. The status of work in this area has recently been reviewed by McCroskey, Ref. 15, and also by Williams, Ref. 8. The importance of boundary layer effects in transonic flow and turbomachinery has been discussed by Tijdeman, Ref. 16, and Gostelow, Ref. 17. For the non-lifting "panel problem" the principal role of the boundary layer is fairly well understood. The dominant effect is an amplitude reduction and phase shift of the inviscid pressure distribution due to the momentum defect in the boundary layer. The corrections are of the order of the ratio of local boundary thickness (or some inverse power of Reynolds number) to the length scale of the surface disturbance. The only significant effect of viscosity in the non-lifting problem is to establish the basic turbulent velocity profile. Beyond this the problem may be considered inviscid and the so-called inviscid parallel shear flow model may be used to calculate the dominant effect of the boundary layer. Theoretical results, both steady and unsteady, correlate very well with experimental data.

The status of the lifting problem is not so clear. First of all, it is difficult to argue that inviscid boundary layer effects are important at all for a lifting airfoil. If corrections are of the same order as in the non-lifting problem (a reasonable supposition) then the thickness of the boundary layer at the trailing edge must be at least as large as the geometric thickness to yield a significant effect. For "well designed" high Reynolds number airfoils, such is not the case. It is for this reason that Dowell and his co-workers (Refs. 9-13) and the author (Refs. 1 and 2) has concentrated primarily on the calculation of boundary layer effects on control surfaces

where the boundary layer thickness may be comparable to the control surface chord.

However, if we examine steady state experimental data on lift curve slope for various airfoil sections (see Section II of the report and Ref. 18), there is a significant discrepancy between the measured and theoretical values. The effect of geometric thickness complicates the situation even further by increasing the lift curve slope. However, at high Reynolds number (1 to 10 million) C_{L_α} is typically 10 to 20% less than the flat plate theoretical value of 2π. It is difficult to imagine anything other than the boundary layer that could reconcile theory and experiment. Further experimental evidence for the importance of the boundary layer in the unsteady problem may be found in the recent work of Satyanarayana and Davis, Ref. 14. For an oscillating airfoil they show that the phase and the amplitude of the surface pressure distribution near the trailing edge departs from the inviscid result (plus "Kutta" condition) for reduced frequency around 0.8. The experimental evidence supports the view that boundary layer effects must be important for the complete airfoil while we are faced with the perplexing result that inviscid parallel flow models cannot predict the magnitude of the correction needed even for the steady state lifting airfoil.

The answer to the dilemma can be found by a closer examination of the steady incompressible problem. Most textbooks that deal with the theory of lift point out that the "Kutta" condition is an emperical condition that accounts for the action of viscosity near the sharp trailing edge. The <u>"total lift"</u> on the airfoil is due to this <u>viscous effect</u>. Indeed, it was one of the great triumphs in the theory of lift that the singular effect of viscosity was recognized. If the <u>total lift</u> is determined by the action of viscosity in the sub-boundary layer then we can see where we have perhaps been wrong in searching for "inviscid boundary layer thickness corrections." The first order of business should be to calculate the way that viscosity establishes

the global circulatory flowfield and the Reynolds number dependence thereof.

In this report, we carry out the above program within the framework of linearized "viscous" thin airfoil theory. Starting with viscous boundary condition, the "Kutta" condition is derived in the limit of high Reynolds number. The correction to the Kutta condition is found to be of order (1/ln Re) , and is much greater than inviscid boundary layer thickness corrections. In fact, the viscous correction competes with and often dominates the effect of geometric thickness. We also derive the unsteady viscous airfoil equation and discuss the singularities of the kernel in the light of recent experimental work in Ref. 14.

II. TWO-DIMENSIONAL VISCOUS STEADY INCOMPRESSIBLE THIN AIRFOIL THEORY

A. Formulation of the Problem

Consider a two-dimensional flat plate airfoil at angle of attack α in an incompressible fluid (see Figure 1).

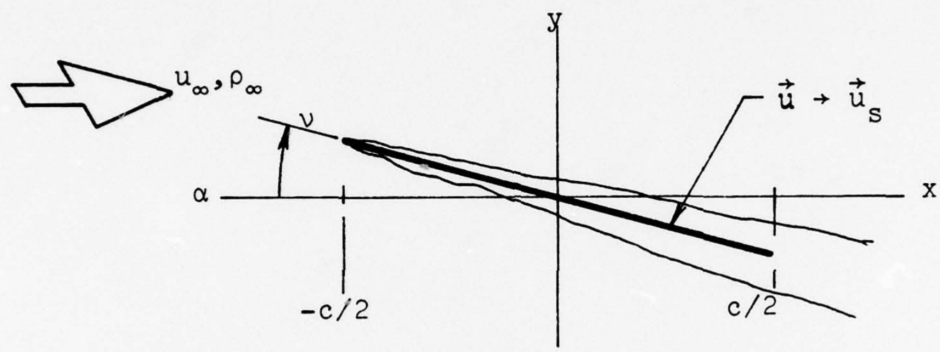

Figure 1. The Lift Problem

The steady state incompressible viscous equations are:

$$\operatorname{div} \vec{u} = 0 \tag{2.1}$$

$$\vec{u} \cdot \frac{\partial \vec{u}}{\partial \vec{x}} = - \operatorname{grad} h + \nu \nabla^2 \vec{u} \tag{2.2}$$

where

$$h = (p - p_\infty)/\rho_\infty \tag{2.3}$$

The viscous boundary condition is

$$\lim_{\vec{x} \to \vec{x}_s} \vec{u}(\vec{x}) = \vec{u}_s(\vec{x}_s) \tag{2.4}$$

where $\vec{x}_s$ denotes the position of the airfoil surface and $\vec{u}_s$ its velocity. Disturbances must also decay at infinity. The problem is to calculate the pressure distribution on the airfoil.

We choose the semichord $c/2$ as the reference length and u_∞ as the reference velocity. For reasons that will be apparent later, we choose the Reynolds number based on the quarter chord as the measure of viscous effects; i.e.,

$$\sigma = \frac{u_\infty c}{4\nu} = \frac{Re}{4} \tag{2.5}$$

where Re is the usual Reynolds number based on the full chord. Next we introduce a perturbation velocity field,

$$\vec{v} = \vec{u} - \vec{i}u_\infty \tag{2.6}$$

where for the moment $|\vec{v}|$ need not be small. The dimensionless problem for $\vec{v}$ becomes:

$$\text{div}\,\vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial x} + \vec{v}\cdot\frac{\partial \vec{v}}{\partial \vec{x}} = -\frac{\partial P}{\partial \vec{x}} + \frac{1}{2\sigma}\nabla^2\vec{v} \tag{2.7}$$

$$P = \frac{p - p_\infty}{\rho_\infty u_\infty^2} \tag{2.8}$$

$$\lim_{\vec{x} \to \vec{x}_s} \vec{v}(\vec{x}) = \vec{u}_s(\vec{x}_s) - \vec{i} \tag{2.9}$$

We denote the pressure on the upper and lower surface by superscripts + and - respectively, and define the usual section lift and moment coefficients:

$$C_L = \frac{L}{q_\infty c} = \frac{1}{q_\infty c}\int_{-c/2}^{c/2}(p^- - p^+)\,dx$$

$$= \int_{-1}^{1}(P^- - P^+)\,dx \qquad (2.10)$$

$$C_M = \frac{M}{q_\infty c^2} = -\frac{1}{q_\infty c^2}\int_{-c/2}^{c/2}(p^- - p^+)x\,dx$$

$$= -\frac{1}{2}\int_{-1}^{1}(P^- - P^+)x\,dx \qquad (2.11)$$

where q_∞ is the dynamic pressure. Clockwise moments are positive and C_M is, by definition, the moment coefficient about the center of the airfoil.

B. Formulation of the Linearized Viscous Problem

The above formulation is exact, although the resulting problem is theoretically intractable. If we invoke the usual "no-slip" boundary condition, the fluid mechanics of the boundary layer and wake are exceedingly complex. The effect on the lift of the retarded flow in these regions has been the principal aim of recent investigations (Refs. 1-13) that start with the so-called inviscid parallel shear flow model. An essential deficiency of the parallel shear flow model (with finite surface velocity) is, however, that one must always make some statement about the nature of the flow at the trailing edge (usually the Kutta condition) or in the wake to fix the inviscid eigensolution (see Section II) of the problem. If the eigensolution is omitted the total lift is exactly zero and the flow at the trailing edge is singular. No matter what details of the inviscid boundary layer and wake flow are included, the situation is the

same.

The first order problem, then, is not to calculate inviscid corrections due to retarded boundary layer and wake flow, but to find out how the "Kutta" or other condition is established via the action of viscosity. Most text books that deal with the subject of lift, point out that "viscosity" is ultimately responsible for the lift. Yet the aerodynamicist seldom includes viscosity explicitly in the lift calculation. The purpose of the following discussion is to show precisely how viscosity is responsible for the "Kutta" condition and the steady state lift on an airfoil and how it leads to the first-order correction due to finite Reynolds number.

Corrections due to inviscid boundary layer effects are known (Refs. 1-13) to be of the order of the boundary layer thickness. To argue that these corrections are small compared to viscous effects, we pose the following boundary conditions. Suppose that we fix a symmetric airfoil at zero angle of attack. Then we "remove" the boundary layer either by symmetric controlled suction or by moving the surfaces of the airfoil at the free stream velocity u_∞. For theoretical purposes the latter approach is favored while the first approach is experimentally more practical. Next, we place the airfoil at a small angle of attack α. The linearized surface velocity,(2.9), becomes,

$$\vec{u}_s = \vec{i}u_\infty - \vec{j}u_\infty\alpha \qquad (2.12)$$

and the boundary conditions on $\vec{v}$ become

$$\left.\begin{aligned} v_x &= u(x,y = 0\pm) = 0 \\ v_y &= v(x,y = 0\pm) = -\alpha \end{aligned}\right\} -1 < x < 1 \qquad (2.13)$$

Also, the linearized fluid equations become

$$\operatorname{div} \vec{v} = 0$$

$$\frac{\partial \vec{v}}{\partial x} = - \frac{\partial P}{\partial \vec{x}} + \frac{1}{2\sigma} \nabla^2 \vec{v} \tag{2.14}$$

We proceed one step further and introduce a streamfunction ψ such that

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = - \frac{\partial \psi}{\partial x} \tag{2.15}$$

Then our linearized boundary value problem becomes

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial y} \left(- \frac{\partial \psi}{\partial x} + \frac{1}{2\sigma} \nabla^2 \psi \right)$$

$$\frac{\partial P}{\partial y} = - \frac{\partial}{\partial x} \left(- \frac{\partial \psi}{\partial x} + \frac{1}{2\sigma} \nabla^2 \psi \right)$$

with

$$\psi_y(x, y = 0\pm) = 0$$

$$\psi_x(x, y = 0\pm) = \alpha \quad , \quad -1 < x < 1 \tag{2.16}$$

We note that the streamfunction satisfies the classical Oseen equation and the pressure is a harmonic function; i.e.,

$$\frac{\partial}{\partial x} \nabla^2 \psi = \frac{1}{2\sigma} \nabla^4 \psi$$

$$\nabla^2 P = 0 \tag{2.17}$$

Also, observe that the quantity $\left(A = -\frac{\partial\psi}{\partial x} + \frac{1}{2\sigma}\nabla^2\psi\right)$ is a harmonic function that is conjugate to P . That is, P and A satisfy the Cauchy-Riemann equations (2.16). Thus, the complex pressure $\mathcal{P} = P + iA$ is an analytic function of the complex variable $z = x + iy$. One may compare these observations with classical potential flow where the potential and streamfunction are conjugate harmonic functions. We do not pursue the development of these concepts in the present report.

The novel feature about (2.16) is the presence of the viscous terms and the associated viscous boundary conditions. In the following section we use these equations in conjunction with the techniques of classical thin airfoil theory to develop an integral equation for the lift distribution on the airfoil.

C. The Viscous Airfoil Equation

It follows from the boundary conditions (2.16) that ψ must be a symmetric function of y so that P must be anti-symmetric in y . We develop a solution of the problem by introducing a distribution of pressure doublets along that part of the real axis occupied by the airfoil, i.e.,

$$P(x,y) = \int_{-1}^{1} Q(\xi)\,\frac{\partial}{\partial y}\ln R\,d\xi \tag{2.18}$$

where

$$R = [(x-\xi)^2 + y^2]^{1/2} \tag{2.19}$$

It is easily shown via standard techniques of classical thin airfoil theory that

$$\begin{aligned} \lim_{y\to 0\pm} P(x,y) &= \pm\pi Q(x) \qquad -1 < x < 1 \\ &= 0 \qquad\qquad |x| > 1 \end{aligned} \tag{2.20}$$

and so

$$Q(x) = - \frac{1}{2\pi} L(x) \tag{2.21}$$

where

$$L(x) = P^{-}(x) - P^{+}(x) \ , \tag{2.22}$$

the lift distribution, is the principal unknown of the problem. Next, we calculate

$$\frac{\partial P}{\partial x} = - \frac{\partial}{\partial x} \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \frac{\partial}{\partial y} \ln R \, d\xi$$

$$= - \frac{\partial}{\partial y} \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \frac{\partial}{\partial x} \ln R \, d\xi$$

$$= - \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial x} - \frac{1}{2\sigma} \nabla^2 \psi \right) \tag{2.23}$$

where we have used the first of equations (2.16). The second of equations (2.16) leads to a similar result with $\frac{\partial}{\partial y}$ replaced by $\frac{\partial}{\partial x}$. Thus, we integrate to obtain

$$\frac{\partial \psi}{\partial x} - \frac{1}{2\sigma} \nabla^2 \psi = \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \frac{\partial}{\partial x} \ln R \, d\xi \tag{2.24}$$

The solution of the last equation can be represented in terms of fundamental solutions. First, we note that the right-hand side of (2.24) is a solution of Laplace's equation, so that

$$\psi = \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \ln R \, d\xi \tag{2.25}$$

is a convenient particular integral. We add to (2.25) a distribution of fundamental solutions of the homogeneous equation

$$\frac{\partial \psi}{\partial x} - \frac{1}{2\sigma} \nabla^2 \psi = 0 \tag{2.26}$$

The appropriate fundamental solution that is symmetric in y is

$$\psi = e^{\sigma x} K_o[\sigma(x^2 + y^2)^{1/2}] \tag{2.27}$$

where K_o is the modified Bessel function of zero order (see Ref. 19). The fundamental solution characterizes the physical process of vorticity diffusion and convection and is needed to satisfy the viscous boundary condition. The natural viscous parameter σ is the Reynolds number based on the quarter chord (see p.11). Thus, we can write the general solution of (2.24) in the form:

$$\psi = \int_{-1}^{1} Q(\xi) e^{\sigma(x-\xi)} K_o(\sigma R)\, d\xi + \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \ln R \, d\xi \tag{2.28}$$

The strength of the viscous solution, $Q(\xi)$, is determined by the viscous boundary condition (see (2.16)); i.e.,

$$\lim_{y \to 0\pm} \frac{\partial \psi}{\partial y} = \lim_{y \to 0\pm} \left\{ \int_{-1}^{1} Q(\xi) e^{\sigma(x-\xi)} \frac{\partial K_o}{\partial y} (\sigma R)\, d\xi \right.$$

$$\left. + \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \frac{\partial \ln R}{\partial y}\, d\xi \right\}$$

$$= \mp \pi Q(x) \pm \frac{L(x)}{2} = 0 \qquad (2.29)$$

so that

$$Q(x) = \frac{L(x)}{2\pi} \qquad (2.30)$$

and

$$\psi = \frac{1}{2\pi} \int_{-1}^{1} L(\xi) \left[\ln R + e^{\sigma(x-\xi)} K_o(\sigma R) \right] d\xi \qquad (2.31)$$

Finally, we apply the second boundary condition (2.16) to obtain the "viscous" airfoil equation;

$$\frac{1}{2\pi} \int_{-1}^{1} L(\xi)\, K(x-\xi)\, d\xi = \alpha \qquad (2.32)$$

where the kernel is given by

$$K(x) = \frac{\partial}{\partial x} \left[\ln |x| + e^{\sigma x} K_o(\sigma |x|) \right]$$

$$= \frac{1}{x} + \sigma e^{\sigma x} \left(K_o(\sigma |x|) - K_1(\sigma |x|) \operatorname{sgn} x \right) \qquad (2.33)$$

If one follows the foregoing steps with viscosity set to zero, the counterpart of (2.32) is the classical airfoil equation,

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{L(\xi)}{x - \xi} \, d\xi = \alpha \tag{2.34}$$

where the integral is a Cauchy principal value.

The importance of viscosity can be illuminated by comparison of the Cauchy kernel in (2.34) with the kernel (2.33). We observe that for $\sigma|x| \to 0$ the kernel (2.33) becomes

$$\begin{aligned} K(x) \underset{\sigma|x| \to 0}{\sim} & \frac{1}{x} + \sigma\left(- \ln \frac{\sigma|x|}{2} - \frac{1}{\sigma x} + O(\sigma|x| \ln \sigma|x|)\right) \\ \sim & - \sigma \ln \frac{\sigma|x|}{2} + O[\sigma^2|x|\ln \sigma|x|] \end{aligned} \tag{2.35}$$

Thus, the viscous kernel only has a weak logarithmic singularity, no matter how large the Reynolds number may be.

Now it is well known that the solution of (2.34) is not unique (see Section III), while the same equation with a logarithmic kernel, i.e.,

$$\frac{1}{2\pi} \int_{-1}^{1} L(\xi) \ln (x - \xi) \, d\xi = \alpha \tag{2.36}$$

has a unique solution. (See Appendix A and Ref. 20.) Thus, we argue that by retaining a finite Reynolds number in (2.33) that we should be able to derive the unique lowest order lift distribution. We show by a moment technique in Section II E that this is indeed the case, and the Kutta condition is the correct lowest order "uniqueness criteria" to apply to (2.34)!

Before we consider the solution of (2.33) we point out another essential difference between the viscous and inviscid kernels. In the inviscid case the kernel has upstream/downstream symmetry. The airfoil does not know whether it is

flying forward or backwards. On the other hand, the kernel (2.33) is highly directional. To illustrate the point, we expand (2.33) for large $\sigma|x|$ to obtain

$$K(x) \sim \frac{1}{x} + e^{-2\sigma|x|}\sqrt{\frac{2\pi}{\sigma|x|}} \quad , \quad \sigma x \to -\infty$$

$$\sim \frac{1}{x}\left(1 - \frac{1}{2}\sqrt{\frac{\pi}{2\sigma|x|}}\right) \quad , \quad \sigma x \to +\infty \tag{2.37}$$

The upstream viscous effect decays exponentially with increasing Reynolds number while it decays algebraically as $1/\sqrt{\sigma|x|}$ in the downstream direction. We shall see that the asymmetry of the viscous kernel leads to removal of the "trailing edge" singularity of the inviscid solution in lowest order in accordance with the Kutta condition.

D. The Inviscid Solution - A Brief Review and Discussion

The solution of the inviscid airfoil equation (2.34) for an arbitrary downwash distribution $\alpha(x)$ can be expressed in closed form; i.e.,

$$L(x) = \frac{A}{\sqrt{1 - x^2}} + \frac{2}{\pi\sqrt{1 - x^2}}\int_{-1}^{1}\frac{\sqrt{1 - \xi^2}}{\xi - x}\,\alpha(\xi)\,d\xi \tag{2.38}$$

where A is an arbitrary constant, the "eigensolution."

For the special case of constant angle of attack it can be verified by direct substitution and use of known relations for Chebyshev polynomials (Ref. 19, p. 783) that

$$L(x) = \frac{A}{\sqrt{1 - x^2}} - \frac{2\alpha x}{\sqrt{1 - x^2}} \tag{2.39}$$

The total lift depends only on the eigensolution; i.e., using (2.10) we get

$$C_L = A \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} = \pi A \tag{2.40}$$

The Kutta condition, stated in the form that "the pressure loading on the trailing edge must vanish," fixes the constant

$$A = 2\alpha \tag{2.41}$$

and we get

$$L(x) = 2\alpha \sqrt{\frac{1 - x}{1 + x}} \tag{2.42}$$

so that

$$C_L = 2\pi\alpha \tag{2.43}$$

The moment about midchord is independent of A ; i.e., from (2.11)

$$C_M = - \frac{1}{2} \int_{-1}^{1} L(x)\ x\ dx$$

$$= \alpha \int_{-1}^{1} \frac{x^2\ dx}{\sqrt{1 - x^2}}$$

$$= \frac{\pi}{2}\ \alpha \tag{2.44}$$

The principal point to be made is that it is the eigensolution and "emperical" uniqueness criteria in the form of the Kutta

condition that determines the total lift.

We also call attention to the sensitive nature of the integration involved in the inviscid eigensolution. Consider the integral

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{d\xi}{(x - \xi)\sqrt{1 - \xi^2}} = 0 \quad , \quad -1 < x < 1 \tag{2.45}$$

For x very close to unity, we note that the integrand in (2.45) is positive over the entire chord, except for the last segment of the trailing edge. Thus, the integral over a vanishingly small segment of the trailing edge must balance the integral over the remainder of the airfoil. Obviously, the same statement can be made about points near the leading edge. With the nonsymmetric viscous kernel,(2.33), it is evident that this delicate balance is destroyed near the edges and the eigensolution cannot satisfy the viscous counterpart of (2.45). This observation further illustrates the very singular way that viscosity establishes the lift on the airfoil.

E. Derivation of the Kutta Condition

The foregoing discussion suggests strongly that the classical Kutta condition is implied by the viscous thin airfoil equation (2.32). While a rigorous theoretical proof is lacking and a detailed numerical solution of the problem has not been carried out, we can show that this is indeed the case by a moment technique. We adopt a trial solution of (2.32) of the form of the inviscid solution (2.39).

$$L(x) = \frac{A_0}{\sqrt{1 - x^2}} + \frac{A_1 x}{\sqrt{1 - x^2}} \tag{2.46}$$

To determine A_0 and A_1 we require that the first two moments of (2.32) be satisfied. That is

$$\frac{1}{2\pi}\int_{-1}^{1} L(\xi)\, F_o(\xi)\, d\xi = 2\alpha$$

$$\frac{1}{2\pi}\int_{-1}^{1} L(\xi)\, F_1(\xi)\, d\xi = 0 \tag{2.47}$$

where

$$F_n(\xi) = \int_{-1}^{1} x^n\, K(x - \xi)\, d: \quad , \qquad n = 0,1 \tag{2.48}$$

and $K(x)$ is given by (2.33). Substitute (2.46) into (2.47) to obtain the following pair of equations for A_0 and A_1 ;

$$\begin{bmatrix} C_{00} & C_{01} \\ & \\ C_{10} & C_{11} \end{bmatrix} \begin{Bmatrix} A_0 \\ \\ A_1 \end{Bmatrix} = \begin{Bmatrix} 2\alpha \\ \\ 0 \end{Bmatrix} \tag{2.49}$$

where

$$C_{nm} = \frac{1}{2\pi}\int_{-1}^{1} \frac{F_n(\xi)\, \xi^m}{\sqrt{1 - \xi^2}}\, d\xi \qquad \begin{array}{l} n = 0,1 \\ m = 0,1 \end{array} \tag{2.50}$$

The solution of (2.49) is

$$A_1 = \frac{2\alpha}{C_{01} - \left(\dfrac{C_{11}}{C_{10}}\right) C_{00}} \tag{2.51}$$

$$A_0 = -2\alpha \frac{(C_{11}/C_{10})}{C_{01} - (C_{11}/C_{10})\, C_{00}} \tag{2.52}$$

In Appendix B we evaluate the coefficients C_{nm} asymptotically for large Reynolds number. The final results are summarized below

$$C_{00} \simeq \frac{1}{4\sqrt{\pi\sigma}} (\ln 64\text{Re} + \gamma) - \frac{\pi}{4\sqrt{\pi\sigma}}$$

$$C_{01} \simeq -1 + \frac{1}{4\sqrt{\pi\sigma}} (\ln 64\text{Re} + \gamma - 4) + \frac{\pi}{4\sqrt{\pi\sigma}}$$

$$C_{10} \simeq \frac{1}{4\sqrt{\pi\sigma}} (\ln 64\text{Re} + \gamma) + \frac{\pi}{4\sqrt{\pi\sigma}} - \frac{2}{\sqrt{\pi\sigma}}$$

$$C_{11} \simeq \frac{1}{4\sqrt{\pi\sigma}} (\ln 64\text{Re} + \gamma - 4) - \frac{\pi}{4\sqrt{\pi\sigma}} + \frac{2}{3\sqrt{\pi\sigma}} \tag{2.53}$$

where $\gamma \simeq .57722$ is Euler's constant and Re is the Reynolds number based on the airfoil chord. Thus we obtain

$$A_0 \sim +2\alpha \left[1 + \left(\frac{20}{3} - 2\pi\right) \Big/ (\ln 64\text{Re} + \gamma) + 0\left(\frac{\ln \text{Re}}{\sqrt{\text{Re}}}\right)\right] \tag{2.54}$$

$$A_1 \sim -2\alpha \left[1 + 0\left(\frac{\ln \text{Re}}{\sqrt{\text{Re}}}\right)\right] \tag{2.55}$$

For infinite Reynolds number we recover the inviscid solution that we obtain with the Kutta condition; i.e.,

$$\lim_{\text{Re} \to \infty} \begin{cases} A_0 = 2\alpha \\ A_1 = -2\alpha \end{cases} \tag{2.56}$$

We note that A_0 and the total lift is determined by the ratio of two viscous terms C_{11} and C_{10} that vanish asymptotically

at the same rate $O(\ln Re/\sqrt{Re})$ for large Reynolds number. The reason that A_0 tends to $+2\alpha$ as opposed to -2α is that the principal effect of the viscous boundary layer is felt downstream. The singular and subtle viscous origin of the lift is thus resolved.

F. Correction to the Lift Curve Slope

The factor in square brackets in (2.54) is the multiplicative Reynolds number correction factor that should be applied to the lift curve slope for the flat plate airfoil; i.e.,

$$C_{L_\alpha} \simeq 2\pi \left(1 + \frac{20/3 - 2\pi}{\ln 64Re + \gamma}\right) \Big/ \text{rad} \qquad (2.57)$$

The significant fact about this result is that the calculated viscous correction factor is $O(1/\ln Re)$, and is large compared to boundary layer thickness corrections that we expect to vary as some inverse power of Reynolds number (e.g., $Re^{-1/2}$ or $Re^{-1/5}$). The positive sign of the correction term in (2.57) is somewhat disturbing in that we would expect viscosity to reduce the lift curve slope. We can argue in favor of a smaller C_{L_α} if we correct for the very important effect of leading edge thickness. For a thin Joukowski airfoil the theoretical lift curve slope is (Ref. 21)

$$C_{L_\alpha} \approx 2\pi \left(1 + \frac{4}{3\sqrt{3}} \tau\right) \Big/ \text{rad} \qquad (2.58)$$

where τ is the thickness to chord ratio. Thus, C_{L_α} increases with increasing geometric thickness. Now, within the framework of our present linearized theory, thickness (actually leading edge curvature) has a very important effect. It removes the square root singularity in the lift distribution at the leading edge. When we evaluate the integrals (see Appendix B) C_{nm} given by (2.53) we notice that the

correction terms in parenthesis result from integration over the trailing edge, while the remaining constant terms result primarily from integration of the singular lift distribution over the leading edge. If finite thickness effects are included, we can argue that the leading edge terms should be omitted in (2.53) or at least reduced. The resulting lift curve slope becomes

$$C_{L_\alpha} \simeq 2\pi \underbrace{\left(1 + \frac{4}{3\sqrt{3}}\tau\right)}_{\substack{\text{Geometric}\\ \text{thickness}\\ \text{factor}}} \underbrace{\left(1 - \frac{4}{\ln 64Re + \gamma}\right)}_{\text{Kutta factor}} \tag{2.59}$$

where we have also included the correction due to geometric thickness. Thus, the effects of thickness and viscosity oppose each other and for typical high Reynolds number thin airfoils the two effects are of comparable order of magnitude.

For example, at a Reynolds number $Re = 10^7$ the viscous correction (Kutta factor) in (2.59) is approximately 0.81 giving a zero thickness lift curve slope of

$$C_{L_\alpha} = 0.0888/\text{deg.} \quad \left(\begin{matrix}\text{Zero}\\ \text{thickness}\end{matrix}\right) \tag{2.60}$$

or approximately a 20% reduction. For an airfoil thickness of 10% the value is increased to

$$C_{L_\alpha} \cong 0.098/\text{deg.} \tag{2.61}$$

The corresponding result for $Re = 10^6$ is

$$C_{L_\alpha} \cong 0.095/\text{deg.} \tag{2.62}$$

only a 3% difference. While the Reynolds number correction is

large the variation with Re is very small at high values of Re because of the logarithmic dependence.

The net effect of Reynolds number and thickness is to reduce the lift curve slope from the classical thin airfoil result $C_{L_\alpha} = .1097/\text{deg.}$ (or $2\pi/\text{rad}$). The values given by (2.61) and (2.62) are typical of measured data for most NACA airfoils at high Reynolds number. In Fig. 2 we compare theory with experimental data (Ref. 18) on a 0015 airfoil. Since the leading edge radius of the four digit series airfoils varies as the square of the thickness, it is not certain what geometric correction should be applied. We have used the 10% thickness results (see (2.61) and (2.62)) for the purpose of comparison. It is certain that the Reynolds number correction is in the correct order of magnitude and has the correct trend for Reynolds numbers greater than 10^6. The universal nature of this correction factor is apparent from the more extensive results given in Table 1. The measured C_{L_α}'s agree remarkably well with our nominal theoretical values for a wide range of airfoils, both cambered and uncambered. We even note the slightly larger values of C_{L_α} at $Re = 10^7$.

G. Concluding Remarks

We have derived a logarithmic Reynolds number correction to the classical Kutta condition and have further shown that it is comparable in magnitude and often exceeds the geometric thickness effect for high Reynolds number thin airfoils. While the basic logarithmic dependence on Reynolds number is believed to be correct, it must be pointed out that the magnitude of the correction term should be calculated more accurately. In particular, the solution of the viscous thin airfoil equation, (2.32), should be calculated in detail for large but finite Reynolds numbers. Also, the leading edge singularity and possibly the trailing edge should be corrected for nonlinear geometric thickness effects. Finally, we remark that the unsteady

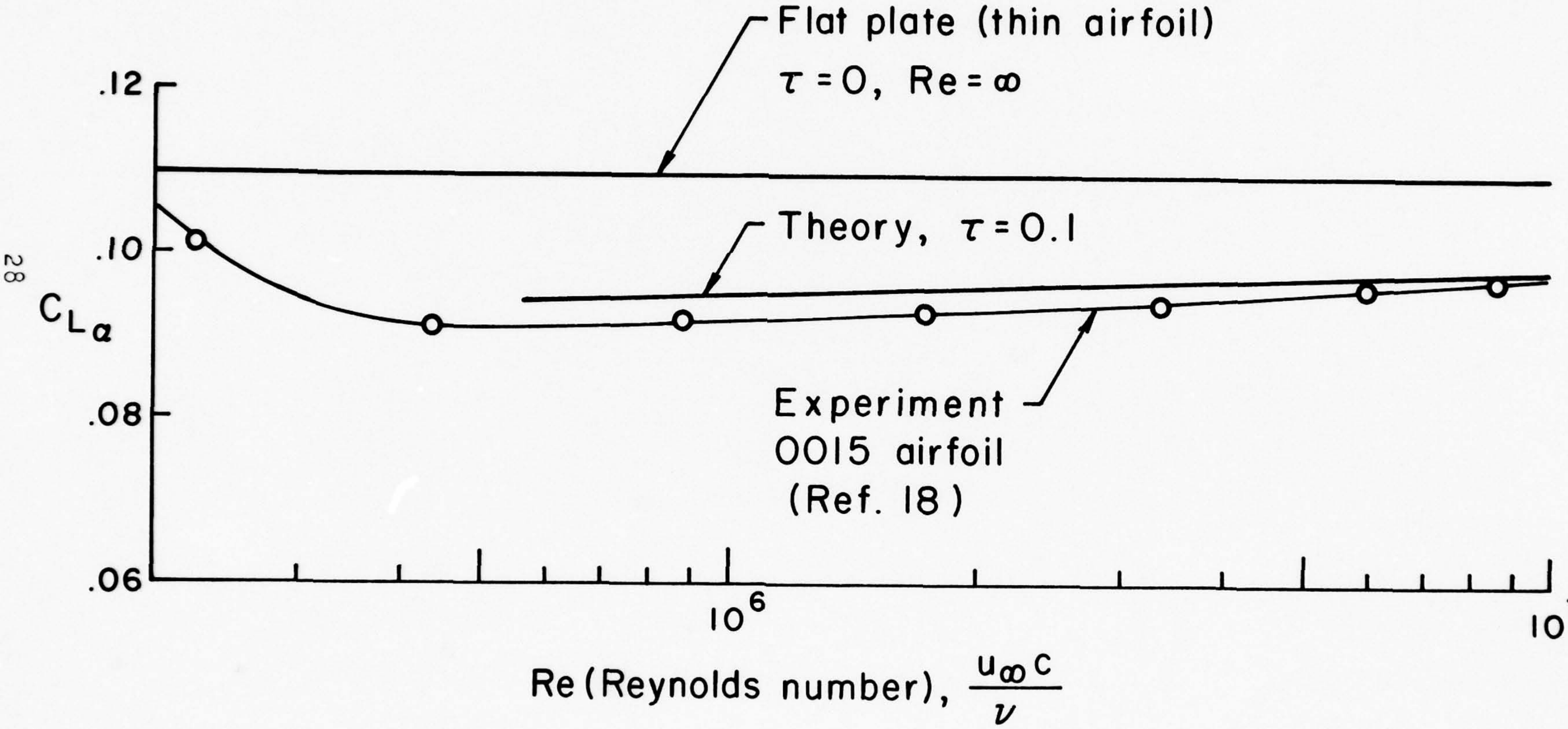

Figure 2. Variation of lift curve slope with Reynolds number for a 0015 airfoil

TABLE 1.

COMPARISON OF EXPERIMENTAL AND THEORETICAL

LIFT CURVE SLOPE FOR VARIOUS AIRFOIL SECTIONS

Experiment Ref. 18	C_{L_α} $Re \simeq 10^6$	$Re \simeq 10^7$
009	.096	.098
0012	.098	.100
0015	.092	.097
2412	.096	.098
23012	.096	.100
23012-33	.094	.097
$2R_212$	.096	.098
4409	.096	.096
4412	.096	.098
4415	.094	.097
6412	.097	.098
Theoretical for 10% thick airfoil	.095	.098

linearized viscous problem should be solved. A formulation and discussion of this problem is given in the following section.

III. EXTENSION TO UNSTEADY INCOMPRESSIBLE FLOW

A. Basic Equations

The basic equations for the unsteady airfoil problem can be derived from the time dependent fluid equations following the steps given in the previous section. To avoid unnecessary repetition of detail, we simply extend the basic results of the previous section given by equation (2.16) We choose the ratio $\frac{c}{2u_\infty}$ as the basic unit of time. Then, the unsteady counterpart of (2.15) and (2.16) can be expressed in the form,

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{D\psi}{Dt} + \frac{1}{2\sigma}\nabla^2\psi\right)$$

$$\frac{\partial P}{\partial y} = -\frac{\partial}{\partial x}\left(-\frac{D\psi}{Dt} + \frac{1}{2\sigma}\nabla^2\psi\right) \tag{3.1}$$

$$\psi_y(x,y = 0\pm,t) = 0$$

$$\psi_x(x,y = 0\pm,t) = -\, w(x,t) \quad -1 < x < 1 \tag{3.2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \quad , \tag{3.3}$$

$$w(x,t) = \frac{Df}{Dt} \quad , \quad -1 < x < 1 \tag{3.4}$$

and $f(x,t)$ is the airfoil deflection mode.

The steps leading to the integrodifferential equation (2.24) of the previous section are also the same, so that

$$\frac{D\psi}{Dt} - \frac{1}{2\sigma}\nabla^2\psi = \frac{1}{2\pi}\int_{-1}^{1} L(\xi,t)\,\frac{\partial}{\partial x}\ln R\,d\xi \tag{3.5}$$

where the basic unknown lift distribution $L(x,t)$ is now a function of time. The boundary conditions (3.2) are to be applied to the solution of (3.5).

B. Review of Inviscid Theory

There is an important fundamental difference between the steady and unsteady problems. In the steady state the vortices shed from the upper and lower surfaces of the airfoil at the trailing edge are equal and opposite in sign, so that a complete cancellation of the vortex wake results. In the inviscid theory, this means that all singularities of the streamfunction are concentrated along the airfoil chord. In the unsteady problem, there is an imbalance in the vortex shedding process at the trailing edge and a time dependent vortex wake is formed. The strength of this wake is a significant factor in determining the total unsteady lift distribution and is again determined completely in the inviscid problem by criteria that we specify at the trailing edge (usually the steady state Kutta condition). To show how the wake affects the kernel of the unsteady airfoil equation we briefly review the classical unsteady inviscid problem (see Ref. 22).

Consider (3.5) with $\sigma = \infty$ and assume simple harmonic motion, i.e.

$$q(x,t) = \left(\mathrm{Re}\ q(x)e^{-ikt}\right) \tag{3.6}$$

where q is any dependent variable and

$$k = \frac{\omega c}{2u_\infty} \tag{3.7}$$

is the usual reduced frequency based on the semi-chord. Then,

$$\frac{\partial\psi}{\partial x} - ik\psi = \frac{1}{2\pi}\int_{-1}^{1} L(\xi)\,\frac{\partial}{\partial x}\ln R\,d\xi \tag{3.8}$$

We integrate (3.8) from upstream infinity to any point x to get

$$\psi(x,y) = \frac{1}{2\pi}\int_{-1}^{1} L(\xi)\,d\xi\cdot\int_{-\infty}^{x} e^{ik(x-s)}\,ds\cdot\frac{\partial}{\partial s}\ln R \tag{3.9}$$

where

$$R^2 = (s - \xi)^2 + y^2 \tag{3.10}$$

Finally, apply the inviscid boundary conditions, (3.2), to obtain the incompressible form of Poisso's integral equation

$$\frac{1}{2\pi}\fint_{-1}^{1} L(\xi)\,K(x - \xi)\,d\xi = -\,w(x) \tag{3.11}$$

where

$$K(x) = \frac{\partial}{\partial x}\fint_{-\infty}^{x} e^{ik(x-s)}\,\frac{ds}{s}$$

$$= \frac{1}{x} + ike^{ikx}\fint_{-\infty}^{x} \frac{e^{-iks}\,ds}{s} \tag{3.12}$$

The Cauchy principal value integral can be expressed in terms of sine and cosine integrals. For present purposes we need

only point out that the second term has a logarithmic singularity. Simply integrate (3.12) by parts to get

$$K(x) = \frac{1}{x} + ik \ln |x| - k^2 e^{ikx} \int_{-\infty}^{x} e^{-iks} \ln |s| \, ds \tag{3.13}$$

The last two terms in (3.13) are due to unsteady vortex shedding in the wake. Far downstream, the wake simply oscillates as e^{ikx} . The absence of wake decay in the inviscid formulation is due to the fact that all wake effects are collapsed into an infintesimally thin sheet of vorticity across which discontinuities in tangential velocity can occur.

The presence of the Cauchy singularity in the kernel means that the unsteady problem does not have a unique solution. By invoking the Kutta condition one can obtain the well known closed form solution of the incompressible problem. If we invoke the condition that no vortex wake shall form, we obtain a different solution of the problem (Refs. 1 and 22). The first solution is consistent with the steady state results obtained in the previous section and is approximately correct for low reduced frequencies. On the other hand, at higher reduced frequencies one might expect the vorticity shed from opposite sides of the airfoil to self annihilate within a very short distance downstream of the trailing edge. Also, the mechanism whereby vorticity is shed into the wake is probably not consistent with the classical steady state Kutta condition as the frequency is increased. The experiments of Satyanarayana and Davis (Ref. 14) show clearly that the classical solution breaks down as the reduced frequency increases towards unity.

Again we expect that an explanation for the discrepancy is to be found in the solution of the viscous problem. While we have not carried out a detailed numerical solution, the basic formulation of the viscous problem and the structure of

the resulting kernel all reinforce the basic idea that linear viscous effects are responsible for the discrepancy.

C. The Unsteady Viscous Airfoil Equation

To develop the viscous airfoil equation we consider the fundamental solution of

$$\frac{\partial\psi}{\partial x} - ik\psi - \frac{1}{2\sigma}\nabla^2\psi = \frac{\partial}{\partial x}\ln r = \frac{x}{r^2} \qquad (3.14)$$

$$r^2 = x^2 + y^2$$

that corresponds to a pressure dipole at the origin. From (3.9) we have the inviscid solution,

$$\psi_o(x,y) = \int_{-\infty}^{x} e^{ik(x-s)}\, ds\, \frac{\partial}{\partial s}\ln R \qquad (3.15)$$

$$R = s^2 + y^2$$

from which we calculate the vorticity

$$-\omega_o = \nabla^2\psi_o = 2\pi\frac{\partial}{\partial x}\left[e^{ikx}H(x)\,\delta(y)\right]$$

$$= 2\pi\left[\delta(x)\,\delta(y) + ike^{ikx}H(x)\,\delta(y)\right] \qquad (3.16)$$

The inviscid vorticity is a point vortex at the origin plus the usual oscillating wake. Let

$$\psi = \psi_o + \psi'$$

in (3.14). Then

$$\frac{\partial\psi'}{\partial x} - ik\psi' - \frac{1}{2\sigma}\nabla^2\psi' = \frac{\pi}{\sigma}\frac{\partial}{\partial x}\left[e^{ikx}H(x)\,\delta(y)\right] \qquad (3.17)$$

The Green's function of the homogeneous equation is

$$G(x,y) = \frac{\sigma}{\pi} e^{\sigma x} K_o(\Omega r)$$
$$\Omega = \sigma (1 - 2ik/\sigma)^{1/2} \tag{3.18}$$

so that

$$\psi' = \iint_{-\infty}^{\infty} d\xi \, d\eta \, e^{\sigma(x-\xi)} K_o(\Omega R) \frac{\partial}{\partial \xi} e^{ik\xi} H(\xi) \, \delta(\eta).$$

$$= \int_{-\infty}^{\infty} d\xi \, e^{\sigma(x-\xi)} \quad K_o(\Omega R) \frac{\partial}{\partial \xi} e^{ik\xi} H(\xi)$$

$$= \int_{0}^{\infty} e^{ik\xi} \frac{\partial}{\partial x} e^{\sigma(x-\xi)} K_o(\Omega R) \, d\xi \tag{3.19}$$

and

$$\psi = \int_{0}^{\infty} e^{ik\xi} \, d\xi \, \frac{\partial}{\partial x} [\ln R + e^{\sigma(x-\xi)} K_o(\Omega R)] \tag{3.20}$$

The unsteady viscous kernel is simply the x derivative of ψ evaluated on the airfoil; i.e.,

$$K(x) = \frac{\partial}{\partial x} \int_{0}^{\infty} e^{ik\xi} \, d\xi \, \frac{\partial}{\partial x} [\ln |x-\xi| + e^{\sigma(x-\xi)} K_o(\Omega|x-\xi|)]$$

$$= \frac{\partial}{\partial x} [\ln |x| + e^{\sigma x} K_o(\Omega|x|)]$$

$$+ ik [\ln |x| + e^{\sigma x} K_o(\Omega|x|)] \tag{3.21}$$

$$- k^2 \int_{0}^{\infty} e^{ik\xi} \, d\xi \, [\ln |x-\xi| + e^{\sigma(x-\xi)} K_o(\Omega|x-\xi|)]$$

where we have integrated by parts to obtain the final result. The terms are grouped in order of decreasing singularity of the inviscid kernel (compare with (3.13)).

We first note that the Cauchy singularity is removed by viscosity for any Reynolds number or frequency. Thus, we expect that the resulting logarithmic singularity will dictate a unique solution as it did in the steady state case. The logarithmic singularity in the unsteady part of the kernel is also removed by viscosity. The relative importance of the in-phase (real) and out-of-phase (imaginary) singularities of the kernel should have a strong effect on the phase of the complex lift distribution. Higher order frequency and Reynolds number effects will result from the last term in (3.21). The effect of viscosity on the unsteady solution is two fold. First, it is responsible for the basic circulation lift with an important Reynolds number correction to the steady state Kutta condition. Second, it affects the phase with which vorticity is shed into the wake. In recent experimental work of Satyanarayana and Davis (Ref. 14) it was observed how the phase and amplitude of the surface pressure departs from inviscid calculation with the steady state Kutta condition applied. For a reduced frequency around 0.8 they note a discrepancy in the phase and amplitude. For reduced frequency near unity there is further steepening of the pressure distribution near the trailing edge. All of the effects they observed are linear, thus indicating that direct viscous action is responsible for the discrepancy. It is believed that the basic physics of these unsteady phenomena is contained in the viscous airfoil equation. Detailed numerical solutions must be carried out and the results compared with the experiments of Satyanarayana and Davis (Ref. 14).

IV. CONCLUSIONS

The principal conclusion of this work is that the classical emperical "Kutta condition" is correct in the limit that Reynolds number approaches infinity and is a direct consequence of the viscous boundary conditions. The correction to the lift curve slope due to finite but large Reynolds number is of order (1/ln Re) and is comparable to and opposes the effect of geometric thickness. The lift curve slope is reduced by approximately 20% for Reynolds number between 1 and 10 million. Comparison with experimental data confirms this calculation. The quantitative effect of viscosity on the circulation lift greatly exceeds the effects of boundary layer thickness or momentum defect.

The unsteady viscous airfoil equation is derived and the singularities of the kernel are discussed. The effect of viscosity is to remove the Cauchy singularity in the inviscid kernel and replace it by a logarithmic singularity. The lowest order frequency dependent terms in the kernel also has a logarithmic singularity that is removed by viscous action. Thus, we expect to see Reynolds number phase variations in the surface load distribution, as the frequency is increased. This conjecture is in accordance with recent experimental data of Satyanarayana and Davis (Ref. 14).

V. RECOMMENDATIONS

With the formulation of the viscous thin airfoil problem, we have opended up an entirely new and important area for research. While we have derived some very significant steady state results, we have only begun to examine the unsteady problems. To further develop this subject the following steps are recommended.

A. Steady State Incompressible Theory

1. Develop numerical or analytic solutions of the viscous thin airfoil equation for large Re .

2. Include geometric thickness effects in the viscous solutions.

3. Compare lift curve slope, center of pressure and pressure distributions with experimental data.

B. Unsteady Incompressible Theory

1. Evaluate the kernel of the viscous unsteady airfoil equation and develop numerical solutions for large Re and reduced frequencies up to order unity.

2. Correct the solution where necessary for geometric thickness effect.

3. Compare results with experimental data of Satyanarayana and Davis (Ref. 14).

C. Unsteady Compressible Theory

Carry out a program similar to A and B with the additional feature of calculating the acoustic far field. Both surface pressure distribution and the acoustic field can be compared with experimental data, e.g., those of Brooks (Ref. 23).

In the design of current generation aircraft, it is often necessary to extrapolate experimental airfoil data in the range $10^6 < Re < 10^7$ up to Reynolds numbers like 10^8 or 10^9. Theoretical results of the type that would be obtained with the above program would be invaluable for establishing intelligent extrapolation procedures. Reliable theoretical procedures would also preclude the necessity for building costly ultra-high Reynolds number test facillities.

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APPENDIX A

A Uniqueness Theorem

Consider the homogeneous integral equation

$$F(x) = \int_{-1}^{1} g(\xi) \ln |x-\xi| \, d\xi = 0 \qquad -1 < x < 1 \tag{A.1}$$

Is there an cigensolution?

Suppose that $g(x)$ is a solution of (A.1). Then the derivative of (A.1) must also vanish; i.e.,

$$\int_{-1}^{1} \frac{g(\xi)}{x-\xi} \, d\xi = 0 \qquad -1 < x < 1 \tag{A.2}$$

Let

$$g(x) = \sum_{n=0}^{\infty} \frac{A_n T_n(x)}{\sqrt{1 - x^2}} \tag{A.3}$$

where the $T_n(x)$ are Chebyshev polynomials of the first kind. They form a complete set of function on the interval $(-1 < x < 1)$ so that (A.3) is a general representation of the solution. Now substitute (A.3) into (A.2) to get

$$\sum_{n=0}^{\infty} A_n \int_{-1}^{1} \frac{T_n(\xi) \, d\xi}{\sqrt{1 - \xi^2}(x - \xi)} = - \pi \sum_{n=1}^{\infty} A_n U_{n-1}(x) = 0 \tag{A.4}$$

where $U_n(x)$ is the Chebyshev polynomial of the second kind. But the $U_n(x)$ are also complete so that all coefficients,

$$A_n = 0 \quad \text{for} \quad n \geq 1 \tag{A.5}$$

Thus,

$$g(x) = \frac{A_o}{\sqrt{1 - x^2}} \qquad \text{(A.6)}$$

is the only solution of (A.2). Is it a solution of (A.1)? Substitute (A.6) into (A.1) and choose $x = 0$ so that

$$A_o \int_{-1}^{1} \frac{\ln |\xi|}{\sqrt{1 - \xi^2}} \, d\xi = 0 \qquad \text{(A.7)}$$

The integral is clearly not zero so that $A_o = 0$ and

$$g(x) = 0 \quad -1 < x < 1 \qquad \text{(A.8)}$$

is the only solution of (A.1)

APPENDIX B

Evaluation of Integrals C_{nm}

Consider the integrals of Section II, (2.50);

$$C_{nm} = \frac{1}{2\pi} \int_{-1}^{1} \frac{F_n(\xi)\ \xi^m}{\sqrt{1 - \xi^2}}\, d\xi \qquad \begin{matrix} n = 0,1 \\ m = 0,1 \end{matrix} \tag{B.1}$$

with

$$F_n(\xi) = \int_{-1}^{1} x^n\ K(x - \xi)\ dx \quad n = 0,1 \tag{B.2}$$

$$K(x) = \frac{\partial}{\partial x} [\ln |x| + e^{\sigma x}\ K_o(\sigma|x|)] \tag{B.3}$$

Evaluate F_n , $n = 0,1$

$$F_o = \int_{-1}^{1} \frac{\partial}{\partial x} [\ln |x-\xi| + e^{\sigma(x-\xi)}\ K_o(\sigma|x-\xi|)]\ dx$$

$$= \ln \left|\frac{1 - \xi}{1 + \xi}\right| + e^{\sigma(1-\xi)}\ K_o[\sigma(1-\xi)]$$

$$- e^{-\sigma(1+\xi)}\ K_o[\sigma(1+\xi)] \tag{B.4}$$

$$F_1 = \int_{-1}^{1} x \frac{\partial}{\partial x} [\ln |x-\xi| + e^{\sigma(x-\xi)}\ K_o(\sigma|x-\xi|)]\ dx$$

$$= \ln (1-\xi^2) + e^{\sigma(1-\xi)}\ K_o[\sigma(1-\xi)]$$

$$+ e^{-\sigma(1+\xi)}\ K_o[\sigma(1+\xi)]$$

$$-\int_{-1}^{1} [\ln |x-\xi| + e^{\sigma(x-\xi)}\ K_o(\sigma|x-\xi|)]\ dx \tag{B.5}$$

The coefficients C_{nm} can be expressed in the form

$$
\begin{aligned}
C_{00} &= A_1 - A_2 \\
C_{01} &= -1 + A_3 - A_4 \\
C_{10} &= A_1 + A_2 - A_5 \\
C_{11} &= A_3 + A_4 - A_6
\end{aligned}
\tag{B.6}
$$

where

$$A_1 = \frac{1}{2\pi}\int_{-1}^{1} \frac{e^{\sigma(1-\xi)}\ K_o[\sigma(1-\xi)]}{\sqrt{1-\xi^2}}\,d\xi \tag{B.7}$$

$$A_2 = \frac{1}{2\pi}\int_{-1}^{1} \frac{e^{-\sigma(1+\xi)}\ K_o[\sigma(1+\xi)]}{\sqrt{1-\xi^2}}\,d\xi \tag{B.8}$$

$$A_3 = \frac{1}{2\pi}\int_{-1}^{1} \frac{\xi e^{\sigma(1-\xi)}\ K_o[\sigma(1-\xi)]}{\sqrt{1-\xi^2}}\,d\xi \tag{B.9}$$

$$A_4 = \frac{1}{2\pi}\int_{-1}^{1} \frac{\xi e^{-\sigma(1-\xi)}\ K_o[\sigma(1+\xi)]}{\sqrt{1-\xi^2}}\,d\xi \tag{B.10}$$

$$A_5 = \frac{1}{2\pi}\int_{-1}^{1} \frac{d\xi}{\sqrt{1-\xi^2}} \int_{-1}^{1} e^{\sigma(x-\xi)}\ K_o(\sigma|x-\xi|)\ dx \tag{B.11}$$

$$A_6 = \frac{1}{2\pi}\int_{-1}^{1} \frac{\xi\ d\xi}{\sqrt{1-\xi^2}} \int_{-1}^{1} e^{\sigma(x-\xi)}\ K_o(\sigma|x-\xi|)\ dx \tag{B.12}$$

We evaluate each of the integrals for large σ . Consider

$$g(\xi) = \int_{-1}^{1} e^{\sigma(x-\xi)} K_o(\sigma|x-\xi|)\, dx \tag{B.13}$$

Let

$$t = \sigma(x-\xi)$$

and write

$$g(\xi) = \frac{1}{\sigma}\int_{o}^{\sigma(1-\xi)} e^{-t} K_o(t)dt + \frac{1}{\sigma}\int_{o}^{\sigma(1-\xi)} e^{t} K_o(t)dt \tag{B.14}$$

For $\sigma \to \infty$ the first term is uniformly of $O(1/\sigma)$ or smaller. The principal contribution to the second term is from the upper limit. We replace K_o by its asymptotic form to get

$$g(\xi) \sim \frac{1}{\sigma}\int_{o}^{\sigma(1-\xi)} \sqrt{\frac{\pi}{2t}} \left(1 + O\left(\frac{1}{t}\right)\right)dt + O\left(\frac{1}{\sigma}\right)$$

$$\sim \sqrt{\frac{2\pi}{\sigma}} \sqrt{1 - \xi} + O\left(\frac{1}{\sigma}\right) \tag{B.15}$$

Substitute the last result into (B.11) and (B.12) to get

$$A_5 = \frac{1}{\sqrt{2\pi\sigma}}\int_{-1}^{1} \frac{d\xi}{\sqrt{1 + \xi}} + O\left(\frac{1}{\sigma}\right)$$

$$= \frac{2}{\sqrt{\pi\sigma}} + O\left(\frac{1}{\sigma}\right) \tag{B.16}$$

$$A_6 = \frac{1}{\sqrt{2\pi\sigma}}\int_{-1}^{1} \frac{\xi\, d\xi}{\sqrt{1 + \xi}} + O\left(\frac{1}{\sigma}\right)$$

$$= -\frac{2}{3} \cdot \frac{1}{\sqrt{\pi\sigma}} + O\left(\frac{1}{\sigma}\right) \tag{B.17}$$

We note that both A_5 and A_6 are primarily a result of integration over the "singular" leading edge lift distribution. The magnitude of both terms will be diminished by the effect of leading edge curvature.

Next we consider the evaluation of integrals A_1 through A_4. We observe that the integrands of A_1 and A_3 are weighted near the trailing edge, while those of A_2 and A_4 are concentrated near the leading edge. Also, the integrands of A_2 and A_4 decay exponentially away from the leading edge while the integrands of A_1 and A_3 decay algebraically. Thus, we expect A_1 and A_3 to be the dominant contribution to the coefficients C_{nm}.

To evaluate A_1 or A_3 we split the interval of integration at $\xi = 1 - \delta$ where $\sigma\delta$ is sufficiently large that $K_o(\sigma\delta)$ can be replaced by its asymptotic expansion. For example, we could choose $\delta = 1/\sigma^{1/2}$. Consider

$$A_1 = \frac{1}{2\pi}\int_{-1}^{1-\delta} + \int_{1-\delta}^{1} d\xi \; \frac{e^{\sigma(1-\xi)} \; K_o[\sigma(1-\xi)]}{\sqrt{1-\xi^2}} \, d\xi$$

$$\cong \frac{1}{2\pi}\int_{-1}^{1-\delta} \sqrt{\frac{\pi}{2\sigma(1-\xi)}} \; \frac{d\xi}{\sqrt{1-\xi^2}} + \frac{1}{2\pi}\int_{1-\delta}^{1} \frac{e^{\sigma(1-\xi)} \; K_o[\sigma(1-\xi)]}{\sqrt{2}\;\sqrt{1-\xi}} \, d\xi + 0\left(\frac{1}{\sigma}\right)$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2\sigma}} \int_{-1}^{1-\delta} \frac{d\xi}{(1-\xi)\;\sqrt{1+\xi}} + \frac{1}{2\pi\;\sqrt{2\sigma}}\int_{0}^{\sigma\delta} e^{\tau}\, K_o(\tau)\,\frac{d\tau}{\sqrt{\tau}} + 0\left(\frac{1}{\sigma}\right) \tag{B.18}$$

The first integral is elementary and the second integral is evaluated in Ref. 23. The final result is

$$A_1 = \frac{1}{4\sqrt{\pi\sigma}}\,(\ln 64 \text{ Re} + \gamma) + 0\left(\frac{1}{\sigma}\right) \tag{B.19}$$

where

$$\gamma = 0.57722 \tag{B.20}$$

is Euler's constant and

$$Re = 4\sigma = \frac{u_\infty c}{\nu} \tag{B.21}$$

is the Reynolds number based on the airfoil chord. To evaluate A_3 we follow the same steps as above. The result is

$$A_3 = \frac{1}{4\sqrt{\pi\sigma}} (\ln 64\, Re + \gamma - 4) + O\left(\frac{1}{\sigma}\right) \tag{B.22}$$

where the 4 is due to the additional power of ξ in the integrand of A_3.

The evaluation of A_2 or A_4 is simple because the only contribution is from the immediate vicinity of the leading edge. Thus we have

$$A_2 = \frac{1}{2\pi} \int_{-1}^{-1+\delta} e^{-\sigma(1+\xi)} \frac{K_o[\sigma(1+\xi)]\, d\xi}{\sqrt{2}\sqrt{1+\xi}} + O(e^{-2\sigma\delta})$$

$$= \frac{1}{2\pi\sqrt{2\sigma}} \int_o^\infty e^{-\tau} K_o(\tau) \frac{d\tau}{\sqrt{\tau}} + O(e^{-2\sigma\delta})$$

$$= \frac{\pi}{4\sqrt{\pi\sigma}} + O(e^{-2\sigma\delta}) \tag{B.23}$$

Similarly

$$A_4 = -\frac{\pi}{4\sqrt{\pi\sigma}} + O(e^{-2\sigma\delta}) \tag{B.24}$$

We emphasize that both A_2 and A_4 are a result of integration over the leading edge. With the integrals A_1 through A_6 we evaluate C_{nm} . The results are summarized below:

Summary

$$C_{00} \simeq \frac{1}{4\sqrt{\pi\sigma}} (\ln 64 \text{ Re} + \gamma) - \frac{\pi}{4\sqrt{\pi\sigma}}$$

$$C_{01} \simeq -1 + \frac{1}{4\sqrt{\pi\sigma}} (\ln 64 \text{ Re} + \gamma - 4) + \frac{\pi}{4\sqrt{\pi\sigma}}$$

$$C_{10} \simeq \frac{1}{4\sqrt{\pi\sigma}} (\ln 64 \text{ Re} + \gamma) + \frac{\pi}{4\sqrt{\pi\sigma}} - \frac{2}{\sqrt{\pi\sigma}}$$

$$C_{11} \simeq \frac{1}{4\sqrt{\pi\sigma}} (\ln 64 \text{ Re} + \gamma - 4) - \frac{\pi}{4\sqrt{\pi\sigma}} + \frac{2}{3\sqrt{\pi\sigma}} \qquad (B.25)$$